NONUNIQUENESS OF THE SOLUTION OF THE SOUND FIELD REPRODUCTION PROBLEM

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ABSTRACT

The sound field reproduction problem is formulated as an inverse problem, in which the reproduction of a target sound field is attempted, in the interior of a given control region, with an array of loudspeakers (referred to as a secondary source distribution). The determination of the loudspeaker gains represents an ill-posed problem. This paper studies under what circumstances the said inverse problem allows for a unique solution.The general solution of the problem is derived, and it is shown that nonuniqueness arises when the wave number is one of the Dirichlet eigenvalues of the control region. It is shown that, when this is not the case, the solution of the problem is unique. Numerical simulations illustrate the effect of nonuniqueness of the solution for the case of spherical secondary source distribution and control region. The case is also studied of the wave number being one of the Dirichlet eigenvalues of the region bounded by the secondary source distribution.

1. INTRODUCTION

The reproduction of a desired sound field using an array of loudspeakers is a problem that has received much attention during the last decade. This can be can be mathematically modeled as an inverse problem, involving a suitable model of the loudspeaker array under consideration and a control region, in which the reproduction of the desired field is attempted. This control region is hereafter identified by the symbol V and, in the case of an interior problem, is contained within the region Λ on the boundary of which the loudspeakers are arranged. In some situations, the two regions Λ and V coincide. In this paper, they are assumed to be two concentric spheres, with radii $R_\Lambda > R_V$.

The study of sound field reproduction as an inverse problem is discussed in the scientific literature and several reproduction techniques have been proposed, such as those described in references [2], [3], [4], [5] and [6], among others. In most cases, the data associated with the problem are a set of measurements of the target field in the control region, or on its boundary. The determination of the required loudspeaker signals from these data is, in general, an ill-posed problem. This implies that the solution might not exist, might not be unique and does in general not depend continuously on the data (instability). In this paper, the problem of nonuniqueness of the solution is addressed.

Portions of the results presented here are also reported in reference [1].

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We assume that the loudspeaker array can be modeled by a continuous distribution of monopole-like secondary sources on the boundary of Λ, and the reproduced field can be therefore mathematically represented by a single layer potential, as explained below. We assume that at least one solution of the problem exists, and we derive an expression for one of the possible solutions.

The Dirichlet problem is discussed, whose uniqueness allows for the sound field control effort to be limited to the boundary of V, rather than to the whole control region, when the operating wave number is not one of the so-called Dirichlet eigenvalues k_n. Otherwise, the solutions of the Dirichlet problem and also of the sound field reproduction problem, as it is formulated here, are not unique.

The proof is provided that the problem under consideration is uniquely solvable when $k \neq k_n$ and, for the case of spherical geometry, it is demonstrated that the opposite is true when k is one of the Dirichlet eigenvalues. It is emphasized that the uniqueness or nonuniqueness of the problem under consideration are determined only by the relation between the wave number k and the geometry of the control region V.

The nonuniqueness problem associated to the Dirichlet eigenvalues (or Neumann eigenvalues for similar problems) is well known and is discussed in the literature dedicated to acoustic radiation problems, boundary element methods and acoustic holography [7],[8], but it has received scarce attention in the literature on sound field reproduction.

In the final part of this paper, the role is studied of the Dirichlet eigenvalues of the region Λ, whose boundary $\partial\Lambda$ correspond to the layer of secondary sources. It is shown that if k coincides with one of these eigenvalues, part of the acoustic energy generated by the loudspeakers is confined to the interior of Λ and does not propagate to its exterior.

2. NOTATION

In what follows, lowercase bold lettering represents vectors. Each vector $\mathbf{x}$ is characterized by a magnitude x and by a unitary vector $\hat{\mathbf{x}}$, identifying the direction of $\mathbf{x}$. It holds that

$$\mathbf{x} = x\,\hat{\mathbf{x}} \tag{1}$$

The relation between Cartesian and polar co-ordinates defining the position vector $\mathbf{x}$ is

$$\mathbf{x} = [x_1,\ x_2,\ x_3] = [x\cos\phi_x\sin\theta_x,\ x\sin\phi_x\sin\theta_x,\ x\cos\theta_x]$$

Given the two sets Λ and V, with boundaries $\partial\Lambda$ and ∂V, respectively, unless differently specified a point on Λ is identified by the vector $\mathbf{y}$, a point on ∂V by the vector $\mathbf{x}$ while any other location in $\mathbb{R}^3$ is identified by the vector $\mathbf{z}$.

Given an operator S acting between two spaces X and Y, its nullspace $N(S)$ is given by the set of all functions $a_N \in Y$ such that $Sa_N = 0$.

The notation $j_\nu(\cdot)$ indicates a spherical Bessel function of order ν, while $h_\nu(\cdot)$ indicates a spherical Hankel functions of the first kind and order ν. The notation $Y_\mu^\nu(\cdot)$ represent a spherical harmonics as defined in, for example, reference [8, p.186].

Given a square integrable function $f(\phi, \theta)$, defined over the unitary sphere, the scalar product $\langle Y_\nu^\mu | f\rangle$ is given by

$$\langle Y_\nu^\mu | f\rangle := \int_0^{2\pi} d\phi \int_0^{\pi} f(\phi,\theta) Y_\nu^\mu(\phi,\theta)^* \sin\theta d\theta \qquad (2)$$

where the symbol $Y_\nu^\mu(\phi,\theta)^*$ indicates the complex conjugate of $Y_\nu^\mu(\phi,\theta)$.

The norm of f is given by

$$||f|| := \left(\int_0^{2\pi} d\phi \int_0^{\pi} |f(\phi,\theta)|^2 \sin\theta d\theta\right)^{\frac{1}{2}} \qquad (3)$$

Given a differentiable function $f(\mathbf{x})$, the normal derivative $\nabla_\mathbf{n} f(\mathbf{x})$ is defined for the spherical geometry considered in this paper by

$$\nabla_\mathbf{n} f(\mathbf{x}) := \frac{\partial f(\mathbf{x})}{\partial x} \qquad (4)$$

that is the radial derivative of the function.

3. FORMULATION AND SOLUTION OF THE PROBLEM

We assume that a continuous distribution of omnidirectional secondary sources is arranged on the boundary $\partial\Lambda$ of a sphere of radius R_Λ. The sound field generated by these secondary sources is given by the single layer potential

$$\begin{aligned}\hat{p}(\mathbf{z}) &= \int_{\partial\Lambda} G(\mathbf{z},\mathbf{y}) a(\mathbf{y}) dS(\mathbf{y}) \qquad (5)\\ &= \int_0^{2\pi} d\phi \int_0^{\pi} G(r_z,\theta_z,\phi_z,r_y,\theta_y,\phi_y) a(\theta_y,\phi_y) R_\Lambda \sin\theta d\theta\\ &\mathbf{x}\in\mathbb{R}^3\end{aligned}$$

where $G(\mathbf{z},\mathbf{y})$ is the free field Green function and $a(\mathbf{y})$ is the source strength density function. A control region V is given, corresponding to a sphere with radius $R_V < R_\Lambda$ and boundaries ∂V. We will refer to the restriction of the single layer potential above to ∂V as the integral operator S. Therefore it holds that

$$\hat{p}(\mathbf{x}) = (Sa)(\mathbf{x}),\ \mathbf{x}\in\partial V \qquad (6)$$

The aim is to reproduce, in the interior of the control region, a target sound field $p(\mathbf{z})$ that satisfies the homogeneous Helmholtz equation

$$\nabla^2 p(\mathbf{z}) + k^2 p(\mathbf{z}) = 0, \quad \mathbf{z}\in\Lambda \qquad (7)$$

The wave number $k = \omega/c$, where ω is the frequency of the sound to be reproduced and c is the speed of sound, assumed to be uniform in $\mathbb{R}^3$.

In order to compute the source strength density function $a(\mathbf{y})$, an inverse problem should be solved. We need to prove if the sound field in the interior of V can be reproduced by limiting the field control effort only on the boundary ∂V of the control region V. In order to do this, we need to introduce the so-called Dirichlet problem.

3.1. The Dirichlet problem

Given the continuous function $f(\mathbf{x})$ defined on ∂V, we seek the solution of the Dirichlet problem

$$\begin{cases}\nabla^2 p(\mathbf{z}) + k^2 p(\mathbf{z}) = 0, & \mathbf{z}\in V\\ p(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x}\in\partial V\end{cases} \qquad (8)$$

where the second equation represents the Dirichlet boundary condition. This is a well known mathematical problem and in references [9] and [10] is shown that it has a unique solution, apart from the case when k is one of the so-called Dirichlet eigenvalues. These are defined as the set of wave numbers k_n such that the problem (8) with homogenous Dirichlet boundary condition $f(\mathbf{x}) = 0$ admits at least one non-trivial solution $p_n(\mathbf{z})$. In physical terms, the problem (8) corresponds to the modal decomposition of the sound field in a cavity with the geometrical shape of V and pressure release boundary conditions ($p(\mathbf{x}) = 0$, $\mathbf{x}\in\partial V$), and the set of wave numbers k_n correspond to the infinite number of resonance frequencies of that cavity. In the case of V being a sphere of radius R_V, it turns out that the set of the Dirichlet eigenvalues k_n are given by those numbers for which $j_n(k_n R_V) = 0$.

In the case of k being one of the Dirichlet eigenvalues, the solution of (8) is not unique. This can be easily proven: assume that $k = k_n$ and $p_n(\mathbf{z})$ is the corresponding eigenfunction (the eigenvalue is assumed to be non degenerate). Given a solution $p(\mathbf{z})$ of (8), the function $\tilde{p}(\mathbf{z}) = p(\mathbf{z}) + \alpha p_n(\mathbf{z})$, for any $\alpha\in\mathbb{C}$, is also a solution. In other words, adding to the solution $p(\mathbf{z})$ the eigenfunction $p_n(\mathbf{z})$, multiplied by an arbitrary complex factor α, results again in a solution of equation (8).

In this case only the Dirichlet boundary condition is not sufficient for solving (8), but it is necessary to impose boundary conditions both on the field and on its gradient (a Cauchy boundary condition). We will see that this nonuniqueness problem of the interior Dirichlet problem has some consequences for the uniqueness of the sound field reproduction problem.

Under the condition that $k \neq k_n$, the arguments above prove that the sound field $p(\mathbf{z})$ in V is uniquely defined by its value on ∂V, and this also implies that if the target sound field is reproduced exactly on ∂V, then it is reproduced exactly also in V.

Using arguments related to the analytical continuation of $\hat{p}(\mathbf{z})$, it can be shown that the condition $\hat{p}(\mathbf{x}) = p(\mathbf{x})$ on ∂V implies that the field is accurately reconstructed also in the region of the space belonging to the interior of Λ and to the exterior of V (namely in $\Lambda\backslash\overline{V}$), provided that $p(\mathbf{z})$ still satisfies the homogeneous Helmholtz equation in that region.

3.2. Solution of the problem

It can be shown (see, for example, reference [5]) that if a solution exists this is given by the solution of the following integral equation

$$p(\mathbf{x}) = (Sa)(\mathbf{x}),\ \ \mathbf{x}\in\partial V \qquad (9)$$

The latter is an integral equation of the first kind, which is well known to represent an ill-posed problem [10]. This implies that its solution might not exist, be unstable or might not be unique. In what follows, we will assume that at least one solution exists, and we study whether only one solution exist or multiple solutions are possible.

Before studying the uniqueness of the problem, we will derive the expression for the solution $a(\mathbf{y})$ (assuming, again, that the latter exists). A more detail discussion about the following derivation can be found, for example, in references [5], [11] or [12]. We will start by expressing the free field Green function by means of spherical harmonics. This is given by [8]

$$G(\mathbf{x},\mathbf{y}) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} ikj_\nu(kx)h_\nu(ky)Y_\mu^\nu(\hat{\mathbf{x}})Y_\nu^\mu(\hat{\mathbf{y}})^* \quad x < y \tag{10}$$

Introducing this formula into equation (9), multiplying both sides of the equation by $Y_\nu^\mu(\hat{\mathbf{x}})^*$, integrating over ∂V and using of the orthogonality relation of the spherical harmonics (see for example [8, p.191], we obtain

$$\langle Y_\nu^\mu|p\rangle = ikj_\nu(kR_V)h_\nu(kR_\Lambda)R_\Lambda\langle Y_\nu^\mu|a\rangle \tag{11}$$

Multiplying both sides of this equation by $Y_\nu^\mu(\hat{\mathbf{y}})$ and applying the completeness relation of the spherical harmonics (see, for example [8, p.191]), we obtain

$$a(\mathbf{y}) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} \frac{\langle Y_\nu^\mu|p\rangle}{ikR_\Lambda j_\nu(kR_V)h_\nu(kR_\Lambda)}Y_\nu^\mu(\hat{\mathbf{y}}) \tag{12}$$

If any $\nu = n$ exists such that $j_n(kx) = 0$, then the corresponding factors should be excluded from the series above, since

$$ikj_\nu(kR_V)h_\nu(kR_\Lambda)R_\Lambda\langle Y_\nu^\mu|a\rangle = 0 \tag{13}$$

independently of $a(\mathbf{y})$.

In the case of a target sound field due to a single omnidirectional virtual source at $\mathbf{q} \notin \Lambda$, the solution is given by combining equations (12) and (10), yielding

$$a(\mathbf{y}) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} \frac{h_\nu(kq)}{R_\Lambda h_\nu(kR_\Lambda)}Y_\nu^\mu(\hat{\mathbf{q}})^*Y_\nu^\mu(\hat{\mathbf{y}}) \tag{14}$$

4. UNIQUENESS OF THE SOLUTION AND DIRICHLET EIGENVALUES

It is shown that the solution of the inverse problem, assuming it exists, is unique if the wave number k is not one of the Dirichlet eigenvalues k_n. On the contrary, if $k = k_n$, the solution of the integral equation is not unique.

The proof of the uniqueness of the solution of (9) is equivalent to the proof of the injectivity of S (see [13] for the definition of injectivity), that is for any two functions $a(\mathbf{y}), a'(\mathbf{y}) \in L^2(\partial\Lambda)$ with $||a-a'|| = 0$ (they are equal in L^2 sense), we have that $Sa \neq Sa'$ [13, p.614]. This is in turn equivalent to the proof that the nullspace of S is trivial, that is $(Sa)(\mathbf{x}) = 0 \rightarrow a(\mathbf{y}) = 0$. This second equivalence can be simply justified as follows: since S is a linear operator, if its nullspace is non-trivial, that is to say if the non-trivial function $a_0(\mathbf{y}) \in L^2(\partial\Lambda)$ exists such that $(Sa_0)(\mathbf{x}) = 0$, then for any function $a(\mathbf{y}) \in L^2(\partial\Lambda)$ it holds that $Sa = S(a + a_0)$, hence S is not injective. Similarly, if S is not injective then two functions $a', a \in L^2(\partial\Lambda)$ exist, with $||a - a'|| \neq 0$, such that $(Sa)(\mathbf{x}) = (Sa')(\mathbf{x})$. Consequently, $\big(S(a - a')\big)(\mathbf{x}) = 0$, which implies that the non-trivial function $(a - a')(\mathbf{y})$ belongs to the nullspace of S (therefore the nullspace of S is non-trivial).

The proof of the uniqueness of the solution is provided by this theorem [1]

THEOREM 4.1. *Given $p(\mathbf{x})$, satisfying equation (7), if the wave number k is not one of the Dirichlet eigenvalues for V, then the solution $a(\mathbf{y})$ of the inverse problem $Sa = p$ is unique.*

Proof of uniqueness theorem

Let the function $a(\mathbf{y})$ belong to the nullspace of S. Then $(Sa)(\mathbf{x}) = 0,\ \forall\mathbf{x} \in \partial V$ and the function

$$u_-(\mathbf{z}) := (Sa)(\mathbf{z}), \quad \mathbf{z} \in V \tag{15}$$

is a solution of the homogeneous interior Dirichlet problem for V. If the wave number k is not one of the Dirichlet eigenvalues for V, then the only solution of the homogeneous interior Dirichlet problem is $u_-(\mathbf{z}) = 0,\ \forall\mathbf{z} \in V$. This implies that, given a subset $W \subset V$, all derivatives of $u(\mathbf{z}),\ \mathbf{z} \in \partial W$ are zero. Hence, for analytical continuation, $(Sa)(\mathbf{z}) = 0,\ \forall\mathbf{z} \in \Lambda$. The continuity of the single layer potential S implies that $(Sa)(\mathbf{z}) = 0,\ \forall\mathbf{z} \in \partial\Lambda$. As a consequence of the uniqueness of the exterior Dirichlet problem [9], we have that $(Sa)(\mathbf{z}) = 0,\ \forall\mathbf{z} \in R^m$. This leads to

$$\lim_{h\to 0} \hat{\mathbf{n}}(\mathbf{z}) \cdot \nabla(Sa)(\mathbf{z} + h\hat{\mathbf{n}}(\mathbf{z})) = 0, \quad \mathbf{z} \in \partial\Lambda \tag{16}$$

for both $h > 0$ and $h < 0$. The jump relation of the single layer potential is given by [10, p.54]

$$\begin{aligned} a(\mathbf{y}) &= \frac{\partial}{\partial x_+}\left[\int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})a(\mathbf{y})dS(\mathbf{y})\right] \\ &- \frac{\partial}{\partial x_-}\left[\int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})a(\mathbf{y})dS(\mathbf{y})\right] \end{aligned} \tag{17}$$

Using this relation, we have that $a(\mathbf{y}) = 0,\ \forall\mathbf{y} \in \partial\Lambda$. This proves that, under the conditions mentioned above, S is injective.

The theorem above also implies that if k is one of the Dirichlet eigenvalues for V, then the solution $a(\mathbf{y})$ is in general not unique. In fact, given the solution (12) in terms of a series of spherical harmonics, any solution of the form $a(\mathbf{y}) + a_0(\mathbf{y})$, where $a_0 \in N(S)$, is also a solution. This case corresponds to the second type of ill-posedness (nonuniqueness) of the integral equation of the first kind (9).

Recalling the discussion presented in Section 3.1, if $k = k_n$ the Dirichlet problem (8) is not uniquely solvable and the knowledge of the pressure profile $p(\mathbf{x}),\ \mathbf{x} \in \partial V$ alone is not enough to determine the field in the interior of V. This suggests that even if the solution of the integral equation (9) is not unique, only one of these solutions is such that the equation

$$\int_{\partial\Lambda} G(\mathbf{z},\mathbf{y})a(\mathbf{y})dS(\mathbf{y}) = p(\mathbf{z}),\ \mathbf{z} \in V \tag{18}$$

is satisfied. In other words, even if the integral equation (9) (note that is different from the formula above) has an infinite number of exact solutions, only one of these solutions allows for the exact reproduction of the target field in the interior (and possibly in the exterior) of the control region V.

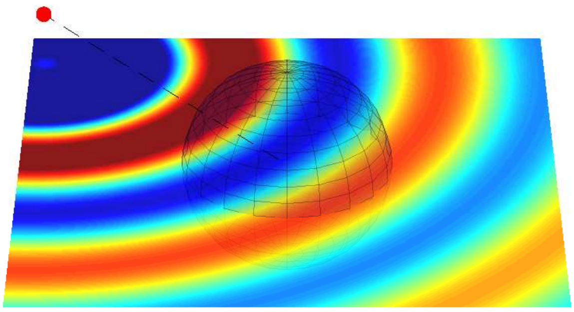

Figure 1: Horizontal cross-section of the field generated by an omnidirectional point source (red dot) located at $[r_q, \theta_q, \phi_q] = [2.5 \text{ m}, 80^\circ, 140^\circ]$. The sphere represents the control boundary ∂V. The wave number is $k = 4 \text{ rad/m}$ and the radius of the sphere is $R_V = 0.7854 \text{ m}$.

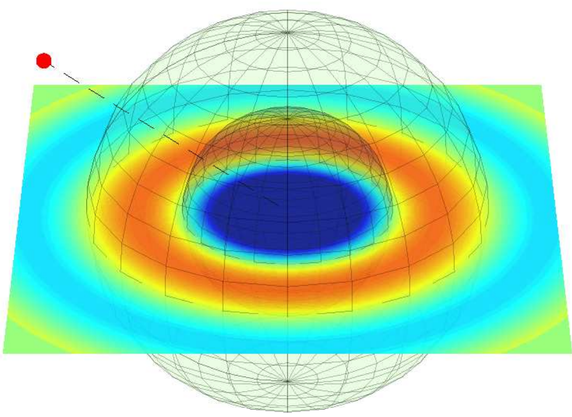

Figure 2: Horizontal cross-section of the field $(SY_0^0)(\mathbf{z})$. The two spheres represent the control boundary ∂V (smaller sphere) and the secondary source layer $\partial\Lambda$. The wave number is $k = 4 \text{ rad/m}$ and the radii of the spheres are $R_V = 0.7854 \text{ m}$ and $R_\Lambda = 1.5 \text{ m}$, respectively.

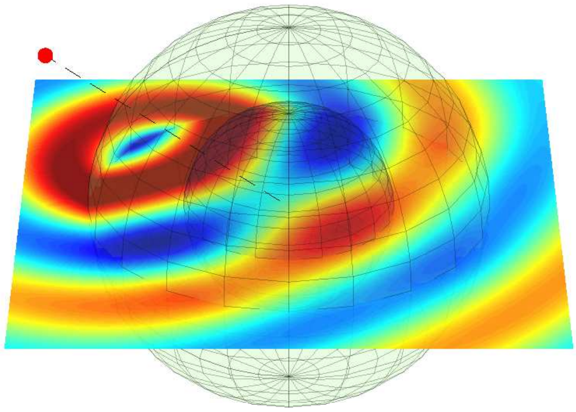

Figure 3: Horizontal cross-section of the reproduced field for a virtual source located at $[r_q, \theta_q, \phi_q] = [2.5 \text{ m}, 80^\circ, 140^\circ]$. The density $a(\mathbf{y})$ was computed with series (12), without the terms with $\nu = 0$. The two spheres represent the control boundary ∂V (smaller sphere) and the secondary source layer $\partial\Lambda$. The wave number is $k = 4 \text{ rad/m}$ and the radii of the spheres are $R_V = 0.7854 \text{ m}$ and $R_\Lambda = 1.5 \text{ m}$, respectively.

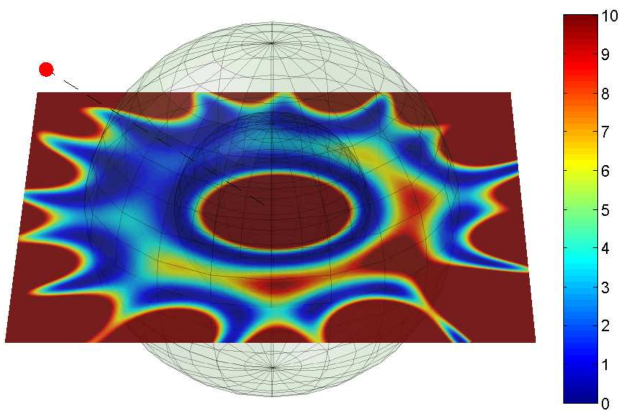

Figure 4: Horizontal cross-section of normalized reproduction error (%) for a virtual source (red dot) located at $[r_q, \theta_q, \phi_q] = [2.5 \text{ m}, 80^\circ, 140^\circ]$. The two spheres represent the control boundary ∂V (smaller sphere) and the secondary source layer $\partial\Lambda$. The wave number is $k = 4 \text{ rad/m}$ and the radii of the spheres are $R_V = 0.7854 \text{ m}$ and $R_\Lambda = 1.5 \text{ m}$, respectively.

4.1. Spherical geometry

We assume now that the wave number k is one of the Dirichlet eigenvalues k_n. Equation (12) represents in this case one of the exact solutions to the integral equation (9), but all solutions of the form

$$a(\mathbf{y}) = \sum_{\substack{\nu=0\\ n\neq\nu}}^{\infty} \sum_{\mu=-\nu}^{\nu} \frac{Y_\nu^\mu(\hat{\mathbf{y}})}{ikR_V^2 R_\Lambda^2 h_\nu^{(1)}(kR_\Lambda) j_\nu(kR_V)} \langle Y_\nu^\mu | p\rangle_{\partial V} + \sum_{m=-n}^{n} \alpha_m Y_n^m(\hat{\mathbf{y}}), \quad \alpha_m \in \mathbb{C} \tag{19}$$

are also correct solutions. Note that the last sum in this expression represents an element in the nullspace of S.

For a single virtual source located at $\mathbf{q} \notin \Lambda$, the exact solution is given by equation (14), while the application of formula (12) to measured data does not give in this case the correct source strength for the reproduction of the target field *in the interior of* V, but only on its boundary. It can be actually seen that (12) does not allow for the computation of the term

$$\frac{h_n(k_n q)}{R_\Lambda^2 h_n(k_n R_\Lambda)} Y_\nu^\mu(\hat{\mathbf{q}})^* Y_\nu^\mu(\hat{\mathbf{y}}) \tag{20}$$

of the series (14) for the given n that identifies the Dirichlet eigenvalue k_n.

It is important to emphasize that this nonuniqueness problem occurs only when there is an *unlucky* combination of the radius R_V and the wave number k. Assuming that the operating frequency is given, only the size of the control region V determines the existence of this problem.

As an example, we assume that the target field, generated by a monopole source in $\mathbf{q}$, is measured on the sphere ∂V with radius R_V such that $j_0(kR_V) = 0$ ($k = 4\,\text{rad/m},\ \mathrm{R_V} = 0.7854\,\text{m}$). This arrangement is illustrated in Figure 1. The pressure profile is given by the series (10). We observe that the first term of the series, $\nu = 0$, equals zero: this implies that the function $Y_0^0(\mathbf{y}),\ \mathbf{y} \in \partial\Lambda$ is in the nullspace of S. Figure 2 shows the field (amplified by a factor 4 for better visualization) given only by the term $\nu = 0$ of series (14), that is

$$p_0(\mathbf{z}) = ikh_0(kq) j_0(kz) Y_0^0(\hat{\mathbf{x}}) Y_0^0(\hat{\mathbf{q}})^* \quad \mathbf{z} \in \mathbb{R}^3,\ z < q \tag{21}$$

It can be noted that ∂V corresponds to a nodal surface of the field, namely $p_0(\mathbf{x}) = 0,\ \mathbf{x} \in \partial V$. Nevertheless $p_0(\mathbf{z}) \neq 0$ in most of the other locations in Λ. A solution $a(\mathbf{y})$ is computed from equation 12 (truncated to the order $N = 7$). Clearly, the series does not include the first term. The orthogonal projections $\langle p_n | p\rangle_{\partial V}$ have been computed via numerical integration on the sphere V. The field $(Sa)(\mathbf{z})$ and the normalized reproduction error are shown in figures 3 and 4, respectively. The normalized reproduction error, between the target field $p(\mathbf{z})$ and the reproduced field $\hat{p}(\mathbf{z})$, is defined by

$$\epsilon_N(\mathbf{z}) = \frac{|\hat{p}(\mathbf{z}) - p(\mathbf{z})|^2}{|p(\mathbf{z})|^2} \tag{22}$$

Not surprisingly, the error approaches zero in the vicinity of ∂V, but it is large in the rest of Λ.

5. DIRICHLET EIGENVALUES OF Λ

It is interesting to note that the Dirichlet (and Neumann) eigenvalues $k_n^{'}$ for the reproduction region Λ do not play any role with respect to the uniqueness of the inverse problem under consideration. It is useful however to notice that the reproduced sound field has the following interesting peculiarity when the wave number is one of the Dirichlet eigenvalues for Λ [1]. Let D_Λ be the linear space of the normal derivatives of the solution of the homogeneous Dirichlet problem for Λ, restricted to $\partial\Lambda$. D_Λ is defined

$$D_\Lambda := \{\nabla_{\mathbf{n}} u(\mathbf{y})|_{\partial\Lambda} : \nabla^2 u(\mathbf{z}) + k^2 u(\mathbf{z}) = 0, \quad \mathbf{z} \in \Lambda, \quad u(\mathbf{y}) = 0,\ \ \mathbf{y} \in \partial\Lambda\} \tag{23}$$

If the secondary sources are driven by a strength function $a_D(\mathbf{y})$, which is an element of the set D_Λ, then we have that

$$(Sa_D)(\mathbf{z}) = 0, \quad \mathbf{z} \in \mathbb{R}^m \backslash \Lambda, \quad a_D(\mathbf{y}) \in D_\Lambda \tag{24}$$

This means that the acoustic field generated by Sa_n vanishes in the exterior of Λ.

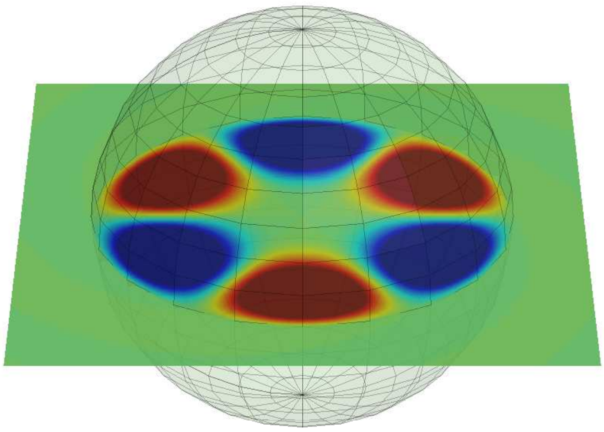

Figure 5: Sound field generated by a continuous distribution of sources on the sphere $\partial\Lambda$. The secondary source strength $a(\mathbf{y})$ is given by the function $Y_5^{-3}(\mathbf{y})/R_\Lambda$, and $j_5(k_n^{'} R_\Lambda) = 0$ ($k_n^{'}$ is one of the Dirichlet eigenvalues of Λ).

For a general geometry, this can be shown using the same arguments used with respect to the uniqueness of the exterior Dirichlet problem [9]. For a spherical geometry, this is simply proven as follows. We assume that $k = k_n^{'}$, one of the Dirichlet eigenvalue for Λ. This implies that $j_n(kR_\Lambda) = 0$. If we assume that the source strength function is given by

$$a(\mathbf{y}) = \sum_{\mu=n}^{n} \alpha_\mu Y_n^\mu(\hat{\mathbf{y}}) \tag{25}$$

for any $\alpha_\mu \in \mathbb{C}$ (it can be easily shown that in this case $a(\mathbf{y}) \in \Psi_\Lambda$). The spherical harmonic expansion of the free field Green function for the case of $x > y$ is similar to (10), but with the role of x and y interchanged. In view of the orthogonality of the spherical harmonics, after some mathematical manipulation it can be shown that

$$\begin{aligned}&\int_{\partial\Lambda} G(\mathbf{z},\mathbf{y})a(\mathbf{y})dS(\mathbf{y}) \qquad (26)\\ =& \sum_{\mu=-n}^{n} ikj_n(kR_\Lambda)h_n(kz)R_\Lambda Y_n^\mu(\hat{\mathbf{y}})\alpha_\mu\\ =&\ 0,\ \ \mathbf{z}\in\mathbb{R}^3/\Lambda\end{aligned}$$

This proves that the reproduced field vanishes in the exterior of Λ.

Figure 5 shows the reproduction of the source strength function $Y_5^{-3}(\mathbf{y})/R_\Lambda$ by a continuous distribution of sources on the sphere $\partial\Lambda = \Omega_{R_\Lambda}$. The wave number k considered is one of the Dirichlet eigenvalues for Λ, more specifically $j_5(kR_\Lambda) = 0$. It can be observed that the field in the region exterior to the sphere is zero, while this is not the case for the field in the interior.

The result shown above can be extended to any general source strength $a(\mathbf{y})$: the field generated by the orthogonal projection of $a(\mathbf{y})$ onto D_V is zero in the exterior of the reproduction region.

6. CONCLUSIONS

It has been shown that problem of determining the strength function of the secondary sources for the reproduction of a desired field in a given control region is uniquely solvable if the wave number under consideration is not one the Dirichlet eigenvalues k_n of the control region V. It has been shown that, when $k = k_n$, the integral operator S is not injective and as a consequence the inverse problem $p = Sa$ is not uniquely solvable. In this case, any linear combination of the functions $a_0(\mathbf{y})$ spanning the nullspace of S can be added to one of the possible solutions $a(\mathbf{y})$ in order to give another solution of the inverse problem (if $k \neq k_n$, the nullspace of S is trivial). Only one of these solutions, however, will in general allow for the accurate reproduction of the desired sound field in the interior of the control region V (and not only on its boundary).

The case has also been considered of k being one of the eigenvalues of Λ. This does not affect the uniqueness of the solution, but the component of the reproduced sound field corresponding to Sa_D is zero in the exterior of Λ, where a_D belongs to the set defined by equation (23).

Future work will explore strategies to overcome the nonuniqueness problem discussed here.

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